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WORMHOLES AND GRAVITONS

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We investigate the effect of wormholes on gravitons. We obtain the two-point function for gravitons in asymptotically flat space in the presence of a wormhole. We show that this result can be reproduced in the theory of gravitons in flat space by the first-order contribution to the propagator due to an effective interaction of the form $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$. In the linear approximation to pure gravity there are no wormhole-induced corrections to the cosmological constant or to Newton's constant.

1. Introduction

There has been interest in the possibility of change of topology in quantum gravity for a long time [1]. Quantum mechanics might allow not only fluctuations in the metric of space-time but also in its topology. Motivated by the success of euclidean path integral techniques in discussions of black hole thermodynamics, Hawking [2] suggested that the formation and evaporation of tiny black holes could be described in that approach by a certain sort of topology change involving baby universes and wormholes. Baby universes are little closed universes (three-surfaces) branching off our large, asymptotically flat universe and wormholes are their four-dimensional histories.

It has been claimed that the effect of wormholes is to cause effective interactions between low-energy fields in flat space-time [2,3]. This is of great interest because these changes of topology therefore modify the coupling constants of any fundamental theory and thus may hinder their predictions. They could contribute to an incertitude over and above that due to the Heisenberg uncertainty principle [2,4]. Moreover, it has been suggested that this mechanism which modifies the coupling constants could explain why the cosmological constant vanishes [5].

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The validity of the wormhole scheme has been challenged by workers (e.g. refs. [5a, 5b]) who point out, amongst other things, that the euclidean path integral approach to quantum gravity has many problems and question the emphasis placed on instanton solutions rather than all wormhole-type geometries. Even within the euclidean path integral formalism the justification for using semiclassical methods seems weak, since, for pure gravity, the dominant contribution comes from wormholes of Planck size whose action is $O(\hbar)$, just the point where it is expected that the differentiable manifold description of space-time will break down. We will not address these problems in this paper, whilst admitting them to be important. We merely make the assumption that there exists a typical wormhole size which is a few orders of magnitude greater than the Planck length, in order that we may use semiclassical approximations.

Hawking [2] has predicted that the effect of wormholes on a conformally invariant scalar field will give rise to all possible local effective interactions. Lyons [6] studied the case of a spin-$\frac{1}{2}$ field and Dowker [7] spin-1. The form of the effective interactions is restricted by the conservation of local charges but not global ones. The purpose of this paper is to extend these results to the spin-2 case. We study pure Einstein gravity and show that gravitons going down wormholes will induce an effective interaction of the type $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$. It is important to notice that there is no correction to the cosmological constant nor to the coefficient of the Ricci scalar curvature R.

At first glance, it is easy to see why only an interaction with four derivatives appears. In $\mathbb{R}^4 - S^4$ (euclidean space with a baby universe at its inner boundary), the lowest graviton mode varies as one over the fourth power of the distance to the baby universe. Thus the Green function behaves as $\propto \int d^4x\, 1/((y_1 - x)^4 (y_2 - x)^4))$ and can only be reproduced by a four-derivative interaction. All other modes decay even faster and have effective interactions with larger numbers of derivatives. In fact it is straightforward to show that the effect of the $n = 3$ inhomogeneous modes of a conformal scalar field going down a wormhole will be reproduced by an effective interaction of the form $(\square\,\phi)^2$ (it is a simple generalisation of the result of Dowker [7], sect. 5). For the graviton case however, the most general covariant second-derivative term that might occur is $\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}$ and it is not a priori obvious which linear combination (i.e. which ratio α/β) will produce the effect of wormholes.

In sect. 2 we define the matrix element that we must calculate using a semiclassical approximation to the path integral. In order to do so we need to know the wormhole wave functions and they are obtained in sect. 3. They are solutions to the Wheeler–DeWitt equation, with boundary conditions such that they correspond to asymptotically flat euclidean geometries. This is in contradistinction to the cosmological case where solutions corresponding to the no-boundary proposal are sought (see ref. [8]). We then calculate the required matrix element in sect. 4. We derive in sect. 5 the same matrix element but from an effective interaction in flat space-time. We conclude in sect. 6 that an interaction of the form $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ reproduces the effect of wormholes. We have also included two appendices describing the harmonics, due to Dowker [9], used to expand tensorial fields on the three-sphere. They are different from the well-known Lifschitz harmonics but they possess a simple transformation rule under rotation of the three-sphere. They are introduced in appendix A and their relation to the Lifschitz ones is demonstrated in appendix B.

2. The matrix element

In this section we describe the transition amplitude between two asymptotically flat universes when a wormhole is present. The process considered is the creation of a graviton at a point x_1 and its annihilation at x_2 in the asymptotically flat region. Without the baby universe this would give the usual Feynman Green function. With the baby universe this will be modified by the possibility of an exchange of gravitons, i.e. the baby universe can absorb a graviton and emit one.

The effect of wormholes can be calculated from a two-point function using the euclidean path integral for quantum gravity. It can be written as

$$\langle 0|\hat{h}_{\mu\nu}(x_1)\hat{h}_{\rho\sigma}(x_2)|\Psi\rangle = \int \mathrm{d}[h_{ij}]\Psi(h_{ij})\int_{\mathscr{C}} \mathrm{d}[g_{\mu\nu}]\hat{h}_{\mu\nu}(x_1)\hat{h}_{\rho\sigma}(x_2)\,\mathrm{e}^{-I_{\mathrm{E}}}, \quad (2.1)$$

where $\mathscr{C}$ is the class of four-geometries which are asymptotically flat (four-dimensionally) and have a compact inner boundary S_b on which the induced metric has the specified value h_{ij}. This sum is weighted by the exponential of the euclidean action I_E. We put hats on $\hat{h}_{\mu\nu}$, the graviton field, to distinguish it from h_{ij} which describes the baby universe. The induced metric is then integrated over, weighted by $\Psi(h_{ij})$, a wormhole wave function.

This is interpreted as the amplitude for a graviton to propagate from x_1 to x_2 in asymptotically flat space-time whilst a baby universe whose quantum state is Ψ branches off.

The matrix element (2.1) will be evaluated using a semiclassical approximation but first we need to describe the wormhole wave functions $\Psi(h_{ij})$ in more detail.

3. The wormhole wave functions

Wormhole wave functions, being the quantum states of closed universes satisfy the Wheeler–DeWitt equation, the quantised hamiltonian constraint of general relativity, with certain boundary conditions.

The action for Einstein gravity is given by

$$I = \frac{m_{\mathrm{P}}^2}{16\pi}\left(\int_{\mathscr{M}} \mathrm{d}^4x\sqrt{g}\,R + 2\int_{\partial\mathscr{M}} \mathrm{d}^3x\sqrt{h}\,(K - K^0)\right) \quad (3.1)$$

where K is the trace of the extrinsic curvature of the boundary and K^0 is the trace of the extrinsic curvature of the boundary when (asymptotically) isometrically embedded in flat space [10, 11]. This latter term is necessary when asymptotically flat spaces are considered in order that the action of the whole of flat space-time be zero.

Writing the line element

$$ds^2 = -(N^2 - N^i N_i)\,dt^2 - 2N_i\,dx^i\,dt + h_{ij}\,dx^i\,dx^j \tag{3.2}$$

and defining the canonical momenta as

$$\pi^{ij} = \frac{\partial \mathscr{L}}{\partial \dot{h}_{ij}}, \tag{3.3}$$

the hamiltonian $H = N_i H^i + N H^0$ can be obtained. The classical equations of motion for N_i and N lead to the momentum constraints

$$H^i = \pi^{ij}{}_{;j} = 0 \tag{3.4}$$

and time reparametrisation constraint

$$H^0 = m_P^{-2} G_{ijkl} \pi^{ij} \pi^{kl} - m_P^2 h^{1/2}\,{}^3R = 0, \tag{3.5}$$

where $G_{ijkl} = \frac{1}{2} h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$.

On quantisation of this theory these classical constraints are translated into constraints on the physical state space. One way to realise them is to take wave functionals of the canonical variables, $\Psi(h_{ij})$, and impose the (functional) differential equations

$$\left(\frac{\delta \Psi}{\delta h_{ij}}\right)_{;j} = 0, \tag{3.6}$$

$$\left(m_P^{-2} G_{ijkl} \frac{\delta^2}{\delta h_{ij}\,\delta h_{kl}} - m_P^2 h^{1/2}\,{}^3R\right)\Psi = 0, \tag{3.7}$$

obtained by making the replacement $\pi^{ij} \to -i\,\delta/\delta h_{ij}$ in eqs. (3.4) and (3.5). Eq. (3.7) is known as the Wheeler–DeWitt equation. We discuss the question of operator ordering ambiguities later in this section.

The Wheeler–DeWitt equation is a differential equation on an infinite-dimensional manifold and is rather difficult to solve exactly. However, we can get an idea of some of its solutions by expanding the gravitational field in a finite-dimensional part that can be treated exactly and some small perturbations around it. We will consider baby universes that are perturbations of the three-sphere. This can be done [12, 13] by expanding the metric (3.2) as

$$N = N_0(t)a(t)\left(1 + g_J{}^N{}_M(t)\,\mathscr{D}^J{}_M{}^N(g)\right),$$

$$N_i = a(t)\left(\sum_J k_{JJ}^{NM}(t)Y^{1JJ}_{mNM}(g) + \sum_{|L-J|=1} j_{LJ}^{NM}(t)Y^{1LJ}_{mNM}(g)\right)\omega_i^m(g),$$

$$h_{ij} = \sigma^2 a^2(t)(\Omega_{ij} + \epsilon_{ij}), \tag{3.8}$$

where Ω_{ij} is the round metric on S^3 and a point on S^3 is written as an element of SU(2), g (see appendix A). The $\{Y^{1LJ}_{mNM}\}$ [9] are the spin-1 hyperspherical harmonics on S^3 and $\{\omega^m{}_i\}$ are a basis of left-invariant one-forms. The $\{\mathscr{D}^J{}_M{}^N\}$ are the elements of the spin-J irreducible representation matrices of SU(2) which, by the Peter–Weyl theorem, form a complete basis for the expansion of scalar fields on S^3.

The perturbation of the three-metric, ϵ_{ij}, is expanded,

$$\epsilon_{ij} = \Omega_{ij}\sum_J \left(\frac{n}{3\pi^2}\right)^{1/2} a_J{}^N{}_M(t)\,\mathscr{D}^J{}_M{}^N(g) + \omega^m{}_i(g)\omega^n{}_j(g)\begin{pmatrix} A & 1 & 1 \\ 2 & m & n \end{pmatrix}\epsilon_A, \tag{3.9}$$

where

$$\begin{aligned}\epsilon_A = &\sum_{L=J}\left(\frac{32}{5}\frac{(n^2-4)}{3(n^2-1)}\right)^{1/2} b_{LJ}^{NM}(t)Y^{2LJ}_{ANM}(g)\\ &+ \sum_{|J-L|=1}\left(\tfrac{32}{5}(n^2-4)\right)^{1/2} c_{LJ}^{NM}(t)Y^{2LJ}_{ANM}(g)\\ &+ \sum_{|J-L|=2} a^{-1}\left(\tfrac{32}{5}\right)^{1/2} d_{LJ}^{NM}(t)Y^{2LJ}_{ANM}(g),\end{aligned} \tag{3.10}$$

and $n = J + L + 1$ in each case. There is a correspondence, demonstrated in appendix B, between the spin-2 hyperspherical spinor harmonics $\{Y^{2LJ}_{ANM}\}$ [9] and the more commonly used tensor harmonics of Lifschitz [14]. For $|L-J| = 2$ they correspond to the transverse traceless harmonics G^n_{ij}, $n = J + L + 1$, for $|L-J| = 1$ to the traceless harmonics S^n_{ij}, $n = J + L + 1$ constructed from the transverse vector harmonics and for $J = L$ to the traceless harmonics P^n_{ij}, $n = 2J + 1$ constructed from the scalar harmonics.

Since the metric perturbation is real we have conditions for each of the sets of coefficients that the complex conjugate of any coefficient equals that coefficient with all its indices reversed in position (upstairs to downstairs or vice versa). Spin-J

indices are raised and lowered using the spin-J metric, $c^{J}{}_{MN} = (-1)^{J-M}\delta_{M,-N} = c^{JMN}$, according to $\phi^M = \phi_N c^{JNM}$ and $V_M = c^{J}{}_{MN}V^N$. The label J will be dropped whenever it is clear what spin is involved.

We will often suppress the M, N indices on the coefficients in the expansion and relabel

$$g_J{}^N{}_M \to g_n, \qquad k_{JJ}^{NM} \to k_n, \qquad a_J{}^N{}_M \to a_n, \qquad n = 2J+1;$$
$$j_{LJ}^{NM} \to j_n, \qquad b_{LJ}^{NM} \to b_n, \qquad c_{LJ}^{NM} \to c_n, \qquad d_{LJ}^{NM} \to d_n, \qquad n = L+J+1. \tag{3.11}$$

Whenever these indices are suppressed in a summation, it is understood that they are summed over as in this example:

$$d_n^2 = \sum_{NM} d_{NM}^{LJ} d_{LJ}^{NM}.$$

This ensures not only the reality of the sums but also their invariance under rotations as is explained below.

From eq. (A.27) the transformation under rotations of each of the coefficients in the expansion can be calculated. For example under $g \to \xi g \eta^{-1}$ where $\xi, \eta \in \mathrm{SU}(2)$

$$d_{NM}^{LJ} \to \mathscr{D}^{L}{}_{N}{}^{N'}(\xi)\, \mathscr{D}^{J}{}_{M}{}^{M'}(\eta)\, d_{N'M'}^{LJ}. \tag{3.12}$$

In general, each index M transforms according to its spin J in the expected way. An expression in which all the indices are summed over "one up and one down" is invariant under rotations.

Calculating the lagrangian and hamiltonian to second order in the perturbations we find that N_0 and the $\{g_n\}$, $\{k_n\}$ and $\{j_n\}$ are independent Lagrange multipliers for the constraints [12],

$$\left[H_{\perp 0} + \sum \left({}^{\mathrm{T}}H^n_{\perp 2} + {}^{\mathrm{V}}H^n_{\perp 2} + {}^{\mathrm{S}}H^n_{\perp 2}\right)\right]\Psi = 0, \tag{3.13a}$$

$$H^n_{\perp 1}\Psi = 0, \qquad {}^{\mathrm{S}}H^n_{|}\Psi = 0, \qquad {}^{\mathrm{V}}H^n_{|}\Psi = 0, \tag{3.13b–d}$$

respectively. The subscripts 0, 1, 2 denote the orders of the quantity in the perturbations; the expansion coefficients and their canonical momenta are denoted by π_{a_n} etc. Eq. (3.13a) is the homogeneous part of $H_\perp \Psi = 0$. $H_{\perp 0}$ is its leading part,

$$H_{\perp 0} = \frac{1}{2a}\left(\pi_a^2 - a^2\right). \tag{3.14}$$

${}^{\mathrm{T,V,S}}H^n_{\perp 2}$ are terms of second order in the scalar perturbations $(a_n, b_n, \pi_{a_n}, \pi_{b_n})$, vector perturbations (c_n, π_{c_n}) and tensor perturbations (d_n, π_{d_n}) respectively. Eq.

(3.13b) is the inhomogeneous part of $H_\perp \Psi = 0$. Eqs. (3.13c) and (3.13d) are the scalar and vector parts of $H_i \Psi = 0$.

The constraints (3.13b–d) are explicitly [12, 13]

$$\left[-\frac{\partial}{\partial a_n} + \tfrac{1}{3}\{(n^2-1)a_n + (n^2-4)b_n\} a \frac{\partial}{\partial a} \right] \Psi = 0,$$

$$\left[-\frac{\partial}{\partial a_n} + \frac{\partial}{\partial b_n} + \left\{ a_n + \frac{4(n^2-4)}{n^2-1} b_n \right\} a \frac{\partial}{\partial a} \right] \Psi = 0,$$

$$\left[-\frac{\partial}{\partial c_n} + 4(n^2-4) c_n a \frac{\partial}{\partial a} \right] \Psi = 0, \tag{3.15}$$

where we have made the representations $\pi_{a_n} \to -i\,\partial/\partial a_n$ etc. It is possible to solve these equations [13] by setting

$$\Psi(a, a_n, b_n, c_n, d_n) = \Psi(\bar{a}, d_n) \tag{3.16a}$$

for

$$\bar{a} = a\left(1 + \tfrac{1}{6} \sum_n (n^2-4)(a_n + b_n)^2 + \tfrac{1}{2} \sum_n a_n^2 - 2 \sum_n \frac{(n^2-4)}{(n^2-1)} b_n^2 - 2 \sum_n (n^2-4) c_n^2 - 2 \sum_n d_n^2 \right), \tag{3.16b}$$

where the coefficients are summed in a covariant manner and the sums over n include summing the two different possibilities for (L, J) where these exist for a given n. The inclusion of the last term is not necessary to solve the constraints but it simplifies eq. (3.13a), the approximation Wheeler–DeWitt equation, considerably. Indeed we find that, in terms of $\bar{a}$ and d_n, it reduces to [13]

$$\left[\left(\frac{\partial^2}{\partial \bar{a}^2} - \bar{a}^2 - \lambda \right) - \sum_n \left(\frac{\partial^2}{\partial d_n^2} - n^2 d_n^2 + \lambda_n \right) \right] \Psi(\bar{a}, d_n) = 0 \tag{3.17}$$

for a particular choice of the factor ordering. The constants λ and λ_n depend on that factor ordering and we discuss them below. The coefficients a_n, b_n and c_n are gauge degrees of freedom and the d_n are the physical, graviton modes.

It seems that the results of our calculation might depend crucially on the choice of factor ordering in (3.17). The choice (3.17) is indeed made to simplify matters – the Wheeler–DeWitt operator becomes a sum of simple harmonic oscillator (SHO) operators. But it turns out that other factor orderings may do just as well as long as we choose an inner product for the wave functions with respect to which the operator is self-adjoint. All we require is a set of wave functions that are orthogonal and labelled by the integers, which label is to be interpreted as the number of gravitons in the wormhole. We choose λ and λ_n so that the ground-state energy of each oscillator is subtracted.

An interesting feature of eq. (3.17) is that it is identical to the approximate Wheeler–DeWitt equation in the case of the conformal scalar [2] or electromagnetic field [7]. It is at first sight surprising that pure gravity gives the same equation because it is not in general conformally invariant. However, it can be shown that for the particular case where only linear gravitons (and possibly conformal matter) are included, the effective energy–momentum tensor for the gravitons is trace free, which implies conformal invariance. Moreover, the absence of graviton creation in a radiation dominated Friedmann–Robertson–Walker universe is related to this conformal invariance [15].

We see from eq. (3.17) that the wormhole wave functions will be sums of products of SHO eigenfunctions – one for the scale factor, a, and one for each mode. The boundary conditions conjectured in ref. [8] for wormhole wave functions are satisfied: they die off at large a and are regular at $a = 0$.

There is, however, a question remaining which is related to the symmetry of the underlying minisuperspace model. We have chosen to expand our fields around a background that is SO(4) invariant. The wave functions we have found ought to be invariant under the action of SO(4). That is, under transformations (3.12) the wave functions should remain the same or at most change by a phase. This is clearly required since the rotated coefficients describe exactly the same field configuration on S^3. One way of achieving the desired invariance is to postulate that the wave functions are functions only of the invariant quantities $d^{LJ}_{NM} d^{NM}_{LJ}$ (only N and M are summed over). We conjecture that this is exactly what would be enforced were we to calculate the second-order momentum constraints that should be imposed due to the linearisation instability of our background [16, 17]. Indeed, in the theory of quantised gravitational perturbations around a background space-time which admits Killing vectors and has closed space-like surfaces, these extra constraints ensure that physical states are invariant under the symmetries generated by those Killing vectors [17].

With the ansatz that the wave functions are functions only of the invariant quantities quadratic in the coefficients, the solutions of eq. (3.17) turn out to be

$$\Psi\left(a, \{d_n^{\pm}\}\right) = H_p(a)\, e^{-a^2/2} \prod_{n, r = \pm} \Psi^n_{2p^r_n}\left(n d^{r2}_n\right), \tag{3.18}$$

where $H_p(a)$ is the Hermite polynomial of order p and the part of the wave function depending on the graviton modes is a product of zero angular momentum eigenfunctions of level $2p^r_n$ of the $(n^2 - 4)$-dimensional SHO. $(n^2 - 4) = (2L + 1)(2J + 1)$ is the number of independent real components of $\{d_n^{+NM}\}$. We have explicitly labelled the two types of harmonic for each n: d_n^{+NM} for $d_{n\,L\,L-2}^{\;N\;M}$ and d_n^{-NM} for $d_{nJ-2J}^{\;N\;M}$ and write d_n^{r2} for $d_n^{rNM} d_{nNM}^{r}$. We interpret $2p_n^+$ ($2p_n^-$) as the number of gravitons of positive (negative) helicity of the mode n in the wormhole and $p = \Sigma_{n,r} 2np_n^r - 1$. In eq. (3.17), $\frac{1}{2}\lambda_n$ is the ground-state energy of the mode n. There also exist solutions which are odd in $\sqrt{d_n^{r2}}$. The interpretation of these solutions is not clear. We would like to be able to say that they are unphysical since we think of a state with an odd number of gravitons as having non-zero spin and wormhole states should be rotationally invariant.

These wave functions are regular everywhere in configuration space and possess a discrete spectrum. If the total energy of the graviton oscillators E is large, the part of the wave function depending on a oscillates rapidly for $a < \sqrt{2E}$. In that region the wave function will be of the form

$$\Psi(\bar{a}) \sim C(\mathrm{e}^{iS} \pm \mathrm{e}^{-iS}), \tag{3.19}$$

where C is the prefactor and S obeys, approximately, the Hamilton–Jacobi equation

$$\left(\frac{\partial S}{\partial \bar{a}}\right)^2 = 2E - \bar{a}^2, \tag{3.20}$$

to which corresponds a family of classical lorentizian geometries

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(2E - (t - t_0)^2\right)\mathrm{d}\Omega_3^2. \tag{3.21}$$

This is a Friedmann–Robertson–Walker (FRW) metric with radiation and maximum radius at $t_{\max} - t_0 = \sqrt{2E}$, the linear gravitons acting as radiation. Such solutions have been investigated by Brill [18].

There exists a wormhole metric which is analytically continued from the FRW radiation dominated universe

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + (2E + \tau^2)\,\mathrm{d}\Omega_3^2 = \left(1 + \frac{E/2}{(x - x_0)^2}\right)^2 \mathrm{d}x^2, \tag{3.22}$$

where $\mathrm{d}x^2$ is the flat space-time metric. It describes a wormhole located at x_0. However, the solution for the classical linear gravitons will not be regular at both asymptotically flat ends. They obey an elliptic differential equation with a strictly positive potential and no regular solutions of such an equation exist on an unbounded manifold. A similar conclusion can be reached for the electromagnetic case and the inhomogeneous part of a scalar or conformal scalar field. In the case of the homogeneous part of a conformal scalar field [2, 19] or an antisymmetric three-index tensor field [3], the condition of strictly positive potential is not satisfied. There, the constant mode is investigated and the potential vanishes, thus regular classical euclidean solutions can exist. As in ref. [20] we will take the point of view here that it is the existence of regular solutions of the Wheeler–DeWitt equation which is important.

4. The semi-classical approximation

Having in hand the wave functions for wormholes, we can now evaluate the matrix element (2.1). This is done for the case of a baby universe containing two gravitons of positive or negative helicity in the lowest ($n=3$) mode. This means that we retain, in the expansion of the gravitational field, only the ten harmonics

$$Y^{2\,2\,0}_{mN0}, \qquad N=\pm 2, \pm 1, 0,$$

$$Y^{2\,0\,2}_{m0N}, \qquad N=\pm 2, \pm 1, 0. \tag{4.1}$$

We relabel their coefficients $D_+{}^N$ and $D_-{}^N$ respectively ($D_\pm$ for short). Here the $\pm$ refers to the helicity state.

The path integral (2.1) is evaluated using a saddle-point approximation. The euclidean action is, up to second order in inhomogeneities and in terms of $\bar{a}$ and d_n,

$$I=\tfrac{1}{2}\int \mathrm{d}t\left(\frac{\tilde{N}}{\bar{a}}\right)\left[-\left(\frac{\bar{a}}{\tilde{N}}\dot{\bar{a}}\right)^2-\bar{a}^2+\sum_n\left(\frac{\bar{a}}{\tilde{N}}\dot{d}_n\right)^2+n^2d_n^2\right], \tag{4.2}$$

where $\tilde{N}=N(1+\Sigma_n d_n^2)$. We drop the overbar on the a and tilde on the N. The equations of motion are, for $N=a$,

$$\ddot{a}=a, \qquad \ddot{D}_\pm=3^2 D_\pm\,. \tag{4.3}$$

We must solve these equations for $a=a_0$ and $D_\pm=D_{\pm 0}$ on the inner boundary (located at t_i say) and $a\,\mathrm{e}^{-t}\to$ constant and $D_\pm\to 0$ as $t\to\infty$. The solutions are

$$a=a_0\,\mathrm{e}^{t-t_i}, \qquad D_\pm=D_{\pm 0}\,\mathrm{e}^{-3(t-t_i)}. \tag{4.4}$$

This classical solution corresponds to flat space outside a three-sphere of radius a_0. The action of this configuration is $\frac{1}{2}(a_0^2+3^2\Sigma D_\pm^2)$.

There is a solution for each position of the wormhole, x_0 and the contribution from each must be counted. For a particular x_0, the gravitational perturbation from flat space is

$$\hat{h}^{x_0}_{ij}(x)=\sigma^4 a_0^3\epsilon^{x_0}_{mn}(x)\,\omega^m_i\big(\widehat{x-x_0}\big)\,\omega^n_j\big(\widehat{x-x_0}\big), \tag{4.5}$$

where

$$\epsilon^{x_0}_{mn}(x) = \begin{pmatrix} 1 & 1 & p \\ m & n & 2 \end{pmatrix} \left(\frac{D^{\ N}_{+0} Y^{220}_{pN0}\left(\widehat{x - x_0}\right)}{(x - x_0)^2} + \frac{D^{\ N}_{-0} Y^{202}_{p0N}\left(\widehat{x - x_0}\right)}{(x - x_0)^2} \right), \qquad (4.6)$$

and $\widehat{x - x_0}$ is the unit vector.

We must refer the components of the metric perturbation to a cartesian frame before we can add up all the saddle-point contributions to eq. (2.1). Altogether we have for the matrix element

$$\begin{aligned}
&\int \mathrm{d}a_0 \prod \mathrm{d}D_0\, \Psi(a_0, D_0) \mathrm{e}^{-a_0^2/2}\, \mathrm{e}^{-3D^2_{+0}/2 - 3D^2_{-0}/2}\, \Delta(a_0)\, a_0^6 \sigma^8 \\
&\quad \times \int \mathrm{d}^4 x\, \Delta^A_\mu(y_1, x) \Delta^B_\nu(y_1, x) \Delta^C_\rho(y_2, x) \Delta^D_\sigma(y_2, x) \\
&\quad \times \frac{1}{(y_1 - x)^2 (y_2 - x)^2} \begin{pmatrix} 1 & 1 & m \\ A & B & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & n \\ C & D & 2 \end{pmatrix} \\
&\quad \times \left(D_{+0}^{\ N} Y_{mN0}^{\ 20}(g_1) + D_{-0}^{\ N} Y_{m0N}^{\ 02}(g_1) \right) \left(D_{+0}^{\ M} Y_{nM0}^{\ 20}(g_2) + D_{-0}^{\ M} Y_{n0M}^{\ 02}(g_2) \right),
\end{aligned} \qquad (4.7)$$

where g_i is the unit vector $\widehat{y_i - x}$, $i = 1, 2$ and $\Delta(a_0)$ is the determinant of small fluctuations about the saddle point. $\Delta^m_\mu(y_1, x)$ is the transformation matrix between the cartesian frame and the spherical basis on S^3 that can be deduced using eqs. (A.5), (A.6), (A.9) and (A.12). It is given by

$$\Delta^m{}_\mu(y_1, x) = \omega^m{}_i(g_1) \Lambda^i{}_\mu(y_1, x)$$

$$= \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} +1 \\ 0 \\ -1 \end{array} & \begin{pmatrix} \frac{-\sqrt{2}}{p^2}\bar{\beta}_p & \frac{-i\sqrt{2}}{p^2}\alpha_p & \frac{\sqrt{2}}{p^2}\alpha_p & \frac{-i\sqrt{2}}{p^2}\bar{\beta}_p \\ \frac{-2i}{p^2}p_4 & \frac{-2i}{p^2}p_3 & \frac{2i}{p^2}p_2 & \frac{2i}{p^2}p_1 \\ \frac{-\sqrt{2}}{p^2}\beta_p & \frac{i\sqrt{2}}{p^2}\bar{\alpha}_p & \frac{\sqrt{2}}{p^2}\bar{\alpha}_p & \frac{i\sqrt{2}}{p^2}\beta_p \end{pmatrix} \end{array}, \qquad (4.8)$$

where

$$\boldsymbol{y}_1 - \boldsymbol{x} = \boldsymbol{p}, \qquad |\boldsymbol{y}_1 - \boldsymbol{x}| = p, \qquad p_1 + ip_4 = \alpha_p, \qquad p_3 + ip_2 = \beta_p. \qquad (4.9)$$

We now use the fact that the measure and wave function are rotationally invariant. For example, consider the integral

$$\int \Pi \, \mathrm{d}D_0 \Psi(a_0, D_0) \mathrm{e}^{-3D_{+0}^2/2 - 3D_{-0}^2/2} D_{+0}^{\ N} D_{+0}^{\ M} .$$

This must be an invariant tensor, symmetric on N and M. The only candidate is a constant multiple of c^{NM}, the spin-2 metric. We see that the constant is non-zero if Ψ describes a wormhole with two positive helicity gravitons and zero if Ψ describes a wormhole with two negative helicity gravitons or more than two gravitons of any helicity (due to orthogonality of the wormhole wave functions). Similar results hold for the other terms. Together with the definitions of the harmonics Y_m^2 these give that for a wormhole containing two negative helicity gravitons the matrix element is proportional to

$$\int \mathrm{d}^4x \, \Delta_\mu^A(y_1, x) \Delta_\nu^B(y_1, x) \Delta_\rho^C(y_2, x) \Delta_\sigma^D(y_2, x)$$
$$\times \begin{pmatrix} 1 & 1 & m \\ A & B & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & n \\ C & D & 2 \end{pmatrix} c_{mn} \frac{1}{(y_1 - x)^2 (y_2 - x)^2} . \tag{4.10}$$

For a wormhole containing negative helicity gravitons it is proportional to

$$\int \mathrm{d}^4x \, \Delta_\mu^A(y_1, x) \Delta_\nu^B(y_1, x) \Delta_\rho^C(y_2, x) \Delta_\sigma^D(y_2, x)$$
$$\times \begin{pmatrix} 1 & 1 & m \\ A & B & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & n \\ C & D & 2 \end{pmatrix} \mathcal{D}^{2\ m'}_{\ n}(g_2^{-1} g_1) c_{m'm} \frac{1}{(y_1 - x)^2 (y_2 - x)^2} , \tag{4.11}$$

with an asymmetry appearing between the two cases. This can be traced back to the fact that we expanded the fields in a *left*-invariant basis.

Using eq. (4.8), a lengthy calculation shows that eq. (4.10) reduces to

$$\begin{aligned} \propto \int \mathrm{d}^4x \Big[& \frac{1}{p^6 q^6} (p \cdot q \delta_{\mu\rho} - p_\rho q_\mu)(p \cdot q \delta_{\nu\sigma} - p_\alpha q_\nu) + (p \cdot q \delta_{\nu\rho} - p_\rho q_\nu)(p \cdot q \delta_{\mu\sigma} - p_\sigma q_\mu) \\ & - \tfrac{2}{3} \left(p^2 \delta_{\mu\nu} - p_\mu p_\nu \right) \left(q^2 \delta_{\rho\sigma} - q_\rho q_\sigma \right) + \left(\varepsilon_{\rho\mu}{}^{\alpha\beta} \varepsilon_{\sigma\nu}{}^{\gamma\delta} + \varepsilon_{\sigma\mu}{}^{\alpha\beta} \varepsilon_{\rho\nu}{}^{\gamma\delta} \right) p_\alpha q_\beta p_\gamma q_\delta \\ & - \left(p_\rho q_\mu \varepsilon_{\sigma\nu}{}^{\alpha\beta} + p_\rho q_\nu \varepsilon_{\sigma\mu}{}^{\alpha\beta} + p_\sigma q_\mu \varepsilon_{\rho\nu}{}^{\alpha\beta} + p_\sigma q_\nu \varepsilon_{\rho\mu}{}^{\alpha\beta} \right) p_\alpha q_\beta \\ & + \left(\delta_{\rho\mu} \varepsilon_{\sigma\nu}{}^{\alpha\beta} + \delta_{\rho\nu} \varepsilon_{\sigma\mu}{}^{\alpha\beta} + \delta_{\sigma\mu} \varepsilon_{\rho\nu}{}^{\alpha\beta} + \delta_{\sigma\nu} \varepsilon_{\rho\mu}{}^{\alpha\beta} \right) p_\alpha q_\beta (p \cdot q) \Big] , \end{aligned} \tag{4.12}$$

where $\boldsymbol{p} = \boldsymbol{y}_1 - \boldsymbol{x}$ and $\boldsymbol{q} = \boldsymbol{y}_2 - \boldsymbol{x}$. Similarly, eq. (4.11) becomes

$$\alpha \int \mathrm{d}^4 x \left[\frac{1}{p^6 q^6} (\boldsymbol{p}\cdot\boldsymbol{q}\delta_{\mu\nu} - p_\rho q_\mu)(\boldsymbol{p}\cdot\boldsymbol{q}\delta_{\nu\sigma} - p_\alpha q_\nu) + (\boldsymbol{p}\cdot\boldsymbol{q}\delta_{\nu\rho} - p_\rho q_\nu)(\boldsymbol{p}\cdot\boldsymbol{q}\delta_{\mu\sigma} - p_\sigma q_\mu) \right.$$

$$- \tfrac{2}{3}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left(q^2\delta_{\rho\sigma} - q_\rho q_\sigma\right) + \left(\varepsilon_{\rho\mu}{}^{\alpha\beta}\varepsilon_{\sigma\nu}{}^{\gamma\delta} + \varepsilon_{\sigma\mu}{}^{\alpha\beta}\varepsilon_{\rho\nu}{}^{\gamma\delta}\right)p_\alpha q_\beta p_\gamma q_\delta$$

$$+\left(p_\rho q_\mu \varepsilon_{\sigma\nu}{}^{\alpha\beta} + p_\rho q_\nu \varepsilon_{\sigma\mu}{}^{\alpha\beta} + p_\sigma q_\mu \varepsilon_{\rho\nu}{}^{\alpha\beta} + p_\sigma q_\nu \epsilon_{\rho\mu}{}^{\alpha\beta}\right)p_\alpha q_\beta$$

$$\left. +\left(\delta_{\rho\mu}\varepsilon_{\sigma\nu}{}^{\alpha\beta} + \delta_{\rho\nu}\varepsilon_{\sigma\mu}{}^{\alpha\beta} + \delta_{\sigma\mu}\varepsilon_{\rho\nu}{}^{\alpha\beta} + \delta_{\sigma\nu}\varepsilon_{\rho\mu}{}^{\alpha\beta}\right)p_\alpha q_\beta(\boldsymbol{p}\cdot\boldsymbol{q}) \right]. \tag{4.13}$$

The integrals differ in their last two lines but these terms vanish by symmetry arguments. The rest of the integral looks as if it diverges at y_1 and y_2 since the integrand $\sim (x - y_1)^{-4}(x - y_2)^{-4}$. However, if the Cauchy principal part is taken it can be shown to be finite. Indeed, by rearranging the integrand into combinations of $A_{\mu\nu}(p)$ and $A_{\mu\nu}(q)$ where $A_{\mu\nu}(x) = \partial_\mu \partial_\nu (x^{-2})$, and using

$$\int \mathrm{d}^4 x A_{\mu\nu}(p) A_{\rho\sigma}(q) = -2\pi^2 \partial_\mu \partial_\nu \partial_\rho \partial_\rho \{\ln(|v|)\} \tag{4.14}$$

where $v = y_1 - y_2$ and

$$A_\mu{}^\mu(x) = -4\pi^2\delta^4(x), \tag{4.15}$$

we can do the integral and obtain for the matrix element

$$I_{\mu\nu\rho\sigma}(y_1 - y_2) \propto \pi^4\left\{-\tfrac{1}{6}\delta_{\mu\nu}\delta_{\rho\sigma} + \tfrac{1}{4}\left(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma}\right)\right\}\delta^4(v)$$

$$+ \pi^2\left\{-\tfrac{1}{24}\left(\delta_{\rho\sigma}A_{\mu\nu}(v) + \delta_{\rho\sigma}A_{\mu\nu}(v)\right)\right.$$

$$\left. + \tfrac{1}{16}\left(\delta_{\mu\rho}A_{\nu\sigma}(v) + \delta_{\nu\rho}A_{\mu\sigma}(v) + \delta_{\mu\sigma}A_{\nu\rho}(v) + \delta_{\nu\sigma}A_{\mu\rho}(v)\right)\right\}$$

$$- \pi^2\left\{\tfrac{1}{24}\partial_\mu \partial_\nu \partial_\rho \partial_\sigma \ln(|v|)\right\}. \tag{4.16}$$

We must now find which effective interaction in flat space-time reproduces this matrix element.

5. The effective interaction

The most general covariant effective interaction with four derivatives has the form [21]

$$\mathscr{L}_{\mathrm{I}} = \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \delta \Box R + \epsilon \varepsilon_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\alpha\beta} R^{\rho\sigma\alpha\beta} . \quad (5.1)$$

But in four dimensions (euclidean), the Euler number is given by

$$\begin{aligned} \chi &= \frac{1}{32\pi^2}\int \mathbf{R}_{ab} \wedge \mathbf{R}_{cd}\varepsilon^{abcd} \\ &= \frac{1}{32\pi^2}\int \mathrm{d}^4x\, g^{1/2}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right). \end{aligned} \quad (5.2)$$

This term has zero functional derivative with respect to $g_{\mu\nu}$ so we can always rewrite the first term in the integral in terms of the other two, thus renormalising α and β in eq. (5.1). The term $\Box R$ is manifestly a covariant total divergence and can be neglected. Finally, the last term in eq. (5.1) also gives a topological invariant, the Pontryagin number,

$$P = \frac{1}{32\pi^2}\int \mathbf{R}_{ab} \wedge \mathbf{R}^{ab} \quad (5.3)$$

and so makes no contribution to the two-point function. The effective interaction therefore reduces to

$$\mathscr{L}_{\mathrm{I}} = \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} . \quad (5.4)$$

There are at least two special cases for α and β [22]. If $3\alpha + \beta = 0$ the interaction is conformally invariant up to a term proportional to the Euler density. Thus, as far as perturbation theory is concerned this interaction term has the same effect as the product of Weyl tensors $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$.

The other interesting special case is $\beta = 0$, where the massive spin-2 excitations disappear [21].

By writing the metric as

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \quad (5.5)$$

eq. (5.4) becomes

$$\begin{aligned} \mathscr{L}_{\mathrm{I}} = {} & \tfrac{1}{2}\beta h^{\alpha\beta}\Box^2 h_{\alpha\beta} + \left(2\alpha + \tfrac{1}{2}\beta\right)h\Box^2 h - (4\alpha + \beta)(\Box h)h^{\alpha\beta}{}_{;\alpha\beta} \\ & + \beta h^{\alpha\beta}{}_{;\beta}\Box h_{\alpha\gamma}{}^{;\gamma} + (2\alpha + \beta)h^{\alpha\beta}{}_{;\alpha\beta}h^{\gamma\delta}{}_{;\gamma\delta} . \end{aligned} \quad (5.6)$$

The unperturbed Green function using the gauge-fixing term

$$-\tfrac{1}{2}\left(h_{\alpha\beta}{}^{;\beta}h^{\alpha\gamma}{}_{;\gamma}+hh_{\alpha\beta}{}^{;\alpha\beta}-\tfrac{1}{4}h\Box h\right) \tag{5.7}$$

is

$$\langle 0|h_{\mu\nu}(y_1)h_{\rho\sigma}(y_2)|0\rangle=\frac{16\pi G}{4\pi^2}\frac{Q_{\mu\nu\rho\sigma}}{(y_1-y_2)^2}, \tag{5.8}$$

where $Q_{\mu\nu\rho\sigma}=\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\nu}\delta_{\rho\sigma}$. The first order contribution to the Green function from $\mathscr{L}$ is

$$\begin{aligned}
I_{\mu\nu\rho\sigma}(y_1,y_2)=&\int d^4x\langle 0|h_{\mu\nu}(y_1)h_{\rho\sigma}(y_2)\left\{\tfrac{1}{2}\beta h^{\alpha\beta}\Box^2h_{\alpha\beta}+\left(2\alpha+\tfrac{1}{2}\beta\right)h\Box^2h\right.\\
&\left.-(4\alpha+\beta)(\Box h)h^{\alpha\beta}{}_{;\alpha\beta}-\beta h^{\alpha\beta}{}_{;\beta\gamma}\Box^2h^{\alpha\gamma}+(2\alpha+\beta)h^{\alpha\beta}{}_{;\alpha\beta}h^{\gamma\delta}{}_{;\gamma\delta}\right\}|0\rangle\\
=&\left(\frac{4G}{\pi}\right)^2\int d^4x\Big[\beta Q_{\mu\nu}{}^{\alpha\beta}Q_{\rho\sigma\alpha\beta}(-4\pi^2)^2\delta^4(x-y_1)\,\delta^4(x-y_2)\\
&+(4\alpha+\beta)Q_{\mu\nu\alpha}{}^{\alpha}Q_{\rho\sigma\beta}{}^{\beta}(-4\pi^2)^2\delta^4(x-y_1)\,\delta^4(x-y_2)\\
&-(4\alpha+\beta)\left[Q_{\mu\nu\alpha}{}^{\alpha}Q_{\rho\sigma\gamma\beta}(-4\pi^2)\,\delta^4(x-y_1)A^{\gamma\beta}(x-y_2)\right.\\
&\left.+Q_{\rho\sigma\alpha}{}^{\alpha}Q_{\mu\nu\gamma\beta}(-4\pi^2)\delta^4(x-y_2)A^{\gamma\beta}(x-y_1)\right]\\
&-2\beta Q_{\mu\nu}{}^{\alpha}{}_{\beta}Q_{\rho\sigma\alpha\gamma}(-4\pi^2)\delta^4(x-y_2)A^{\gamma\beta}(x-y_1)\\
&+2(2\alpha+\beta)Q_{\mu\nu\alpha\beta}Q_{\rho\sigma\gamma\delta}A^{\alpha\beta}(x-y_1)A^{\gamma\delta}(x-y_2)\Big].
\end{aligned} \tag{5.9}$$

This matrix element can be calculated:

$$\begin{aligned}
\left(\frac{\pi}{4G}\right)^2 I_{\mu\nu\rho\sigma}(y_1,y_2)=&(4\pi^2)^2\left[2\beta(\delta_{\mu\nu}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho})+4\alpha\delta_{\mu\nu}\delta_{\rho\sigma}\right]\delta^4(v)\\
&+(4\pi^2)\left[-4(2\alpha+\beta)(\delta_{\mu\nu}A_{\rho\sigma}+\delta_{\rho\sigma}A_{\mu\nu})\right.\\
&\left.+2\beta(\delta_{\mu\rho}A_{\nu\sigma}+\delta_{\mu\sigma}A_{\nu\rho}+\delta_{\nu\rho}A_{\mu\sigma}+\delta_{\nu\rho}A_{\mu\sigma})\right]\\
&-16\pi^2(2\alpha+\beta)\,\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\{\ln(v)\},
\end{aligned} \tag{5.10}$$

with $A_{\mu\nu}\equiv A_{\mu\nu}(v)$.

It is easy to verify that the result (4.16) is equivalent to (5.10) for $3\alpha = -\beta$ and thus the effective interaction due to wormholes containing two gravitons can be written as the square of the Weyl tensor with arbitrary amounts of the Euler density, Pontryagin density and $\Box R$. It is rather surprising that the result in sect. 4 turns out to be in the same gauge as the one used here.

An interesting observation is that the terms in eqs. (4.12) and (4.13) which have opposite signs and vanish on integration are *exactly* those that appear when the contribution to the two-point function from an effective interaction of the form (5.3) is calculated. Though this is zero in the case of the two-point function, it will be important for higher n-point functions. Indeed it seems that wormholes will cause parity violating interactions such as $R^2 R^*_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ where ${}^*R^{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R^{\rho\sigma}{}_{\alpha\beta}$.

6. Conclusion

We have investigated the effect of wormholes on linear gravitons by calculating the Green function in the presence of a wormhole and two gravitons in an asymptotically flat region. We first derived the approximate Wheeler–DeWitt equation for a FRW model filled with linear gravitons. We showed that there exist solutions which represent quantum wormholes. The two point function was calculated by a semiclassical approximation to a euclidean path integral, using the rotational invariance of the baby universe state. Finally we have shown that the two-point function can be interpreted as the contribution to the flat space-time Green function due to an effective interaction. Specifically, wormholes containing two gravitons will induce an interaction of the form $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$. It is interesting to note that there will be no modification, in the linear approximation, to Newton's constant nor to the cosmological constant. The dependence of both these constants on wormholes could come from renormalisation by matter loops which themselves depend on the wormhole parameters.

This result is interesting in itself but we might speculate that this could have important consequences for the initial state of the universe. Indeed Penrose [23] some years ago proposed that the universe started in a state with zero Weyl tensor which would imply a smooth initial state. If there exists an argument similar to the one of Coleman to induce a particular value of the coupling constant for the Weyl tensor, it might be possible to understand the origin of the large scale homogeneity in the universe. This is under present investigation.

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Appendix A

In this appendix we describe the hyperspherical harmonics due to Dowker [9] used to expand spinor fields of arbitrary spin on the three-sphere. These are harmonics whose transformation properties under rotations are transparent. The more common harmonics on S^3, those of Lifschitz [14], do not have such a direct connection to the (double cover of the) symmetry group, $SU(2) \times SU(2)$.

The hyperspherical spinor harmonics are built using the fact that S^3 is the group manifold of SU(2). We can thus associate a point x_μ on the sphere to an element g of the group using the identification

$$(x_1, x_2, x_3, x_4) \to \begin{pmatrix} x_1 + ix_4 & x_3 + ix_2 \\ -x_3 + ix_2 & x_1 - ix_4 \end{pmatrix}, \tag{A.1}$$

where the $x_\mu \in \mathbb{R}$ and are constrained by $\Sigma x_\mu^2 = 1$, thus defining a point on S^3. We could also have chosen the Euler angle parametrisation of SU(2):

$$x_1 + ix_4 = \cos\tfrac{1}{2}\theta\, e^{i(\psi+\phi)/2}, \qquad x_3 + ix_2 = \sin\tfrac{1}{2}\theta\, e^{i(\psi-\phi)/2}. \tag{A.2}$$

We are interested in the group of symmetries of S^3 which is SO(4). This group is isomorphic to $SU(2) \times SU(2)/Z_2$. Under a rotation,

$$g \to g' = R(g) = \xi g \eta^{-1}, \qquad \xi, \eta \in SU(2). \tag{A.3}$$

(Note that replacing $\xi \to -\xi$ and $\eta^{-1} \to -\eta^{-1}$ gives rise to the same rotation and hence the division by Z_2.)

SO(4) is the group which leaves the quadratic form

$$\sum x_\mu^2 = 1 \tag{A.4}$$

invariant. It is generated by the six infinitesimal rotation operators [24]

$$D_{\mu\nu} = -D_{\nu\mu} = -i\left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} \right). \tag{A.5}$$

By introducing the new operators

$$\begin{aligned} M &= \tfrac{1}{2}(\hat{x}_1(D_{43} + D_{12}) + \hat{x}_2(D_{31} + D_{42}) + \hat{x}_3(D_{14} + D_{32})), \\ N &= \tfrac{1}{2}(\hat{x}_1(D_{43} - D_{12}) + \hat{x}_2(D_{31} - D_{42}) + \hat{x}_3(D_{14} - D_{32})), \end{aligned} \tag{A.6}$$

we obtain the following commutation relations:

$$[M,N]=0, \qquad [M_a,M_b]=\epsilon_{ab}{}^{c}M_c, \qquad [N_a,N_b]=\epsilon_{ab}{}^{c}N_c. \tag{A.7}$$

This clearly shows that locally the group is a product of two SU(2)'s. The group is characterised by two Casimir operators

$$C_1=(M^2+N^2), \qquad C_2=(M^2-N^2). \tag{A.8}$$

In order to expand fields on the three-sphere we choose the left-invariant basis of vectors

$$e_a=-iN_a, \qquad a=1,2,3 \tag{A.9}$$

which have the commutation relations

$$[e_a,e_b]=-\epsilon_{ab}{}^{c}e_c. \tag{A.10}$$

The dual basis of one-forms, $\{\omega^b\}$, gives a natural bi-invariant metric on S^3:

$$\mathbf{g}=\delta_{ab}\omega^a\otimes\omega^b. \tag{A.11}$$

This metric is related by a factor of 4 to the metric, $\mathbf{\Omega}$, induced on S^3 by the flat euclidean metric on $\mathbb{R}^4$:

$$4\mathbf{\Omega}=g. \tag{A.12}$$

We will also define a "spherical" basis $\{e_m\}$,

$$e_{\pm}=\pm\frac{i}{\sqrt{2}}(e_1\mp ie_2), \qquad e_0=-ie_3 \tag{A.13}$$

and its dual

$$\omega^{\pm}=\mp\frac{i}{\sqrt{2}}(\omega^1\pm i\omega^2), \qquad \omega^0=i\omega^3. \tag{A.14}$$

(A corresponding basis for right invariant vectors $\{\hat{e}_m\}$ and one-forms $\{\hat{\omega}_m\}$ could equally be used.)

Suppose we want to expand a co-tensor field T on S^3. We may expand it using the basis $\{\omega^m\}$ or $\{dx^i\}$ (where $\{x^i\}$ are some coordinates on S^3),

$$T(g)=T_{m\ldots n}\omega^m(g)\otimes\ldots\otimes\omega^n(g)=T_{i\ldots j}\,dx^i(g)\otimes\ldots\otimes dx^j(g), \tag{A.15}$$

$$T_{m\ldots n}(g)=(T(g),e_m(g)\otimes\ldots\otimes e_n(g))=T_{i\ldots j}\,e_m{}^{i}\ldots e_n{}^{j}(g). \tag{A.16}$$

Under a general coordinate transformation $g \to g'$, $T_{i \ldots j}$ transforms as a co-tensor

$$T_{i \ldots j}(g) \to T'_{i \ldots j}(g') = \frac{\partial g^k}{\partial g'^i} \cdots \frac{\partial g^l}{\partial g'^j} T_{k \ldots l}(g), \tag{A.17}$$

$\omega^m{}_i$ transforms as the components of three covector fields

$$\omega^m{}_i(g) \to \omega'^m{}_i(g') = \frac{\partial g^j}{\partial g'^i} \omega^m{}_j(g) \tag{A.18}$$

and $T_{m \ldots n}$ transforms as a set of scalars.

However, when $g \to g'$ is a *symmetry* transformation, i.e. $g' = \xi g \eta^{-1}$ we have

$$\omega_i'^m(g') = \mathscr{D}^1{}_n{}^m(\eta) \omega^n{}_i(g'), \tag{A.19}$$

remembering that $\{\omega^m\}$ are left-invariant. The $\mathscr{D}^1{}_n{}^m(\eta)$ are the spin-1 irrep matrices for SU(2). This is analogous to Pauli's theorem that there exists a 4×4 matrix, $S(\Lambda)$, for every Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu$ such that $\Lambda^\mu{}_\nu \gamma^\nu = S(\Lambda)^{-1} \gamma^\mu S(\Lambda)$. Then

$$T'_{i \ldots j}(g') = T'_{m \ldots n}(g') \omega^m{}_i(g') \ldots \omega^n{}_j(g'),$$

$$T'_{m \ldots n}(g') = \mathscr{D}^1{}_m{}^p(\eta) \ldots \mathscr{D}^1{}_n{}^q(\eta) T_{p \ldots q}(g). \tag{A.20}$$

Thus, by keeping the basis one-forms fixed under rotations (analogously, keeping the gamma matrices fixed under Lorentz transformations) we can instead say that the $T_{m \ldots n}$ transform as a tensor product of the (left spin-0, right spin-1) representation of the symmetry group SU(2) × SU(2). In the case where the tensor field is a one-form then we say that T_m is a "right" spin-1 spinor. For a symmetric two index tensor field the product representation decomposes to give "right" scalars and spin-2 spinors. They can be obtained by contracting a 3*J*-symbol and two basis one-forms as follows

$$\begin{pmatrix} j & 1 & 1 \\ p & m & n \end{pmatrix} \omega^m(g) \otimes \omega^n(g), \quad j = 2, 0. \tag{A.21}$$

Notice that when $j = 1$ the antisymmetry of the 3*J*-symbol gives zero.

Such an object can be expanded in terms of the spin-*j* right spinor hyperspherical harmonics [9]

$$Y^{jLJ}_{mNM}(g) = \sqrt{\frac{(2L+1)(2J+1)}{16\pi^2}} \mathscr{D}^L{}_N{}^{N'}(g) \begin{pmatrix} L & J & j \\ N' & M & m \end{pmatrix}. \tag{A.22}$$

These Y's are eigenspinors of a complete set of commuting observables formed from the generators of the symmetry group $SU(2) \times SU(2)$ acting on objects with right spin-j. These are

$$j^2, L^2, J^2, J_3, \bar{L}_3 \tag{A.23}$$

with eigenvalues $j(j+1), L(L+1), J(J+1), M, N$ respectively. j is right spin, L is right orbital angular momentum, $J = j + L$ is total right angular momentum and $\bar{L}$ is left angular momentum ($L^2 = \bar{L}^2$, so $\bar{L}^2$ is not needed, though it also commutes with everything).

In appendix B we relate these harmonics with right spin-$j = 0, 1, 2$ to the scalar, vector and tensor harmonics of Lifschitz.

Spin-j indices are raised and lowered using a spin-j metric,

$$c^j_{mn} = c^{jmn} = (-1)^{j-m}\,\delta_{m,-n}, \tag{A.24}$$

according to

$$\phi^m = \phi_n c^{jnm} \quad \text{and} \quad V_m = c^j{}_{mn} V^n. \tag{A.25}$$

The Y's are normalised such that

$$\int \mathrm{d}g\, Y^{mL'J'}_{j\,N'M'}(g)\, Y^{j\,L\,J}_{mNM}(g) = \delta_{LL'}\delta_{JJ'}c_{NN'}c_{MM'}, \tag{A.26}$$

where $\mathrm{d}g$ is the volume element of SU(2) from the metric g so that $\int \mathrm{d}g = 16\pi^2$. They also have the important property that under rotations they behave in the following way:

$$Y^{j\,L\,J}_{mNM}(\xi g \eta^{-1}) = \mathscr{D}^j{}_m{}^{m'}(\eta)\,\mathscr{D}^L{}_N{}^{N'}(\xi)\,\mathscr{D}^J{}_M{}^{M'}(\eta)\,Y^{j\,L\,J}_{m'N'M'}(g). \tag{A.27}$$

We could also have chosen a right-invariant basis of vectors and developed left-spinor harmonics to expand out the corresponding spinor fields.

Appendix B

In this appendix we relate the spinor harmonics of appendix A to the usual Lifschitz [14] tensor harmonics.

Other work dealing with harmonics relevant to Friedmann–Robertson–Walker space-times include the ones of Gerlach and Sengupta [25], Gibbons [26], Hu [27], and Jantzen [28].

The Lifschitz harmonics are eigenfunctions of the laplacian based on the following parametrisation of the three-sphere:

$$ds^2 = d^2\chi + \sin^2\chi\left(d^2\theta + \sin^2\theta\, d^2\phi\right). \tag{B.1}$$

They are Gegenbauer polynomials in χ and the usual spherical harmonics in θ and ϕ. The Peter–Weyl theorem shows that the spin-0 spinor harmonics are linear combinations of the Lifschitz scalar harmonics.

We now demonstrate the correspondence between the Lifschitz vector and tensor harmonics and the spin-1 and 2 spinor harmonics. The Lifschitz vector harmonics are $\{(S_i)^n_{lm}\}$, and $\{(P_i)^n_{lm}\}$ which satisfy

$$\nabla^2 S_i^{(n)} = -(n^2-2)S_i^{(n)}, \qquad \nabla^2 P_i^{(n)} = -(n^2-3)P_i^{(n)}, \tag{B.2}$$

where ∇^2 is the covariant vector laplacian on S^3 formed with the metric $\mathbf{\Omega}$.

The Lifschitz tensor harmonics are $\{(G_{ij})^n_{lm}\}$, $\{(S_{ij})^n_{lm}\}$ and $\{(P_{ij})^n_{lm}\}$ which satisfy

$$\begin{aligned}\nabla^2 G_{ij}^{(n)} &= -(n^2-3)G_{ij}^{(n)},\\ \nabla^2 S_{ij}^{(n)} &= -(n^2-6)S_{ij}^{(n)},\\ \nabla^2 P_{ij}^{(n)} &= -(n^2-7)P_{ij}^{(n)},\end{aligned} \tag{B.3}$$

where ∇^2 is the covariant tensor laplacian on S^3.

Claim 1. Each $S_i^{(n)}$ is a linear combination of the $\omega^m{}_i Y^{1LJ}_{m\cdot\cdot}$ with $L = J = (n-1)/2$ and each $P_i^{(n)}$ is a linear combination of the $\omega^m{}_i Y^{1LJ}_{m\cdot\cdot}$ with $|L-J| = 1$ and $J + L + 1 = n$.

Proof. We first note that indices in the first half of the alphabet, up to and including l, are with respect to a coordinate on S^3, whereas indices from m onwards are with respect to the spherical basis.

$$\begin{aligned}\nabla^2\left\{\omega^m{}_k Y^1_m\right\} &= \Omega^{ij}\,\nabla_i\,\nabla_j\left\{\omega^m{}_k Y^1_m\right\}\\ &= \Omega^{ij}\left\{\tfrac{1}{4}\varepsilon_{jk}{}^l\varepsilon_{il}{}^h\omega^m{}_h Y^1_m + \varepsilon_{jk}{}^l\omega^m{}_l\,\nabla_i Y^1_m + \omega^m{}_k\,\nabla_i\,\nabla_j Y^1_M\right\}\end{aligned} \tag{B.4}$$

since $\nabla_i\omega^n{}_k = \frac{1}{2}\varepsilon_{ik}{}^l\omega^n{}_l$, where $\varepsilon_{ik}{}^l$ is the epsilon tensor of the metric $\mathbf{g}$ in the coordinate basis. We obtain

$$\nabla^2\left(\omega^m{}_k Y^1_m\right) = \omega^m{}_k\left(-2 - 4\boldsymbol{j}\cdot\boldsymbol{L} - 4\boldsymbol{L}^2\right)Y^1_m \tag{B.5}$$

using the fact that ∇_i acting on a spinor, such as Y^1_m, is just a partial derivative, ∂_i, and

$$e_m{}^j \partial_j = iL_m, \qquad [j_m^{(1)}]_n{}^l = -i\varepsilon_{mn}{}^l. \tag{B.6}$$

We have $j \cdot L = \frac{1}{2}(J^2 - L^2 - j^2)$ and, since the Y's are eigenspinors of J^2, L^2 and j^2, the eigenvalues of ∇^2 are

$$\begin{aligned} &-(L+J+1)^2+3 \quad \text{when } L=J, \\ &-(L+J+1)^2+2 \quad \text{when } |L-J|=1. \end{aligned} \tag{B.7}$$

Since both sets of harmonics are complete this completes the proof.

Claim 2. The $G_{ij}^{(n)}$, $S_{ij}^{(n)}$ and $P_{ij}^{(n)}$ are linear combinations of the

$$\omega^m{}_i \omega^n{}_j \begin{pmatrix} \hat{m} & 1 & 1 \\ 2 & m & n \end{pmatrix} Y^{2LJ}_{\hat{m}NM}$$

with $|L-J|=2$, $|L-J|=1$ and $L=J$ respectively and $n=L+J+1$ in each case.

Proof. As before we calculate

$$\nabla^2 \left\{ \omega^m{}_i \omega^n{}_j \begin{pmatrix} \hat{m} & 1 & 1 \\ 2 & m & n \end{pmatrix} Y^{2LJ}_{\hat{m}NM} \right\} \tag{B.8}$$

using

$$g^{mn} \begin{pmatrix} \hat{m} & 1 & 1 \\ 2 & m & n \end{pmatrix} = 0, \tag{B.9}$$

$$\frac{1}{2}\begin{pmatrix} \hat{m} & 1 & 1 \\ 2 & m & n \end{pmatrix} \varepsilon_{pq}{}^m + \frac{1}{2}\begin{pmatrix} \hat{m} & 1 & 1 \\ 2 & m & p \end{pmatrix} \varepsilon_{nq}{}^m = -\frac{i}{2}\begin{pmatrix} \hat{n} & 1 & 1 \\ 2 & p & n \end{pmatrix} [j_q^{(2)}]_{\hat{n}}{}^{\hat{m}}, \tag{B.10}$$

where $[j_q^{(2)}]_{\hat{n}}{}^{\hat{m}}$ is the spin-2 representation matrix of the angular momentum operator j_q. We find that

$$\begin{aligned} \nabla^2 \left\{ \omega^m{}_i \omega^n{}_i \begin{pmatrix} \hat{m} & 1 & 1 \\ 2 & m & n \end{pmatrix} Y^{2LJ}_{\hat{m}NM} \right\} &= \omega^m{}_i \omega^n{}_j \begin{pmatrix} \hat{m} & 1 & 1 \\ 2 & m & n \end{pmatrix} [-6 - 4j \cdot L - 4L^2] Y^{2LJ}_{\hat{m}NM} \\ &= [6 - 2J(J+1) - 2L(L+1)] \omega^m{}_i \omega^n{}_j \begin{pmatrix} \hat{m} & 1 & 1 \\ 2 & m & n \end{pmatrix} Y^{2LJ}_{\hat{m}NM}. \end{aligned} \tag{B.11}$$

TABLE B.1

Relations between scalar, vector and tensor harmonics of Lifschitz and spin-0, -1 and -2 spinor harmonics. The correspondence is that each Lifschitz harmonic is a linear combination of the quantities involving spinor harmonics opposite to it in the table

Harmonics	Lifschitz	Spinor	
Scalar	Q^{n}_{lm}	Y^{0KK}_{0NM}	$n=2K+1$
Vector (transverse)	$(S_i^{o,e})^{n}_{lm}$	$\left\{\begin{matrix} \omega^{m}{}_{i}Y^{1\,K\,K-1}_{m\,N\,M} \\ \omega^{m}{}_{i}Y^{1\,K-1\,K}_{m\,N\,M} \end{matrix}\right\}$	$n=2K$
Vector (scalar derived)	$(P_i)^{n}_{lm}$	$\omega^{m}{}_{i}Y^{1KK}_{mNM}$	$n=2K+1$
Tensor (transverse traceless)	$(G_{ij}^{o,e})^{n}_{lm}$	$\left\{\begin{matrix} \omega^{A}{}_{i}\omega^{B}{}_{j}\begin{pmatrix} m & 1 & 1 \\ 2 & A & B \end{pmatrix}Y^{2\,K-2\,K}_{m\,N\,M} \\ \omega^{A}{}_{i}\omega^{B}{}_{j}\begin{pmatrix} m & 1 & 1 \\ 2 & A & B \end{pmatrix}Y^{2\,K\,K-2}_{m\,N\,M} \end{matrix}\right\}$	$n=2K-1$
Tensor (vector derived)	$(S_{ij}^{o,e})^{n}_{lm}$	$\left\{\begin{matrix} \omega^{A}{}_{i}\omega^{B}{}_{j}\begin{pmatrix} m & 1 & 1 \\ 2 & A & B \end{pmatrix}Y^{2\,K-1\,K}_{m\,N\,M} \\ \omega^{A}{}_{i}\omega^{B}{}_{j}\begin{pmatrix} m & 1 & 1 \\ 2 & A & B \end{pmatrix}Y^{2\,K\,K-1}_{m\,N\,M} \end{matrix}\right\}$	$n=2K$
Tensor (scalar derived)	$(P_{ij})^{n}_{lm}$	$\omega^{A}{}_{i}\omega^{B}{}_{j}\begin{pmatrix} m & 1 & 1 \\ 2 & A & B \end{pmatrix}Y^{2KK}_{mNM}$	$n=2K+1$
Tensor (trace)	$(Q_{ij})^{n}_{lm}$	$\omega^{A}{}_{i}\omega^{B}{}_{j}c_{AB}Y^{0KK}_{0NM}$	$n=2K+1$

The eigenvalues are

$$-(L+J+1)^2+3 \quad \text{when } |L-J|=2,$$

$$-(L+J+1)^2+6 \quad \text{when } |L-J|=1,$$

$$-(L+J+1)^2+7 \quad \text{when } L+J. \tag{B.12}$$

This verifies the correspondence.

The comparison between Lifschitz and spinor harmonics for the scalar, vector and tensor harmonics is summarised in table B.1.

References

[1] J.A. Wheeler, Geometrodynamics (Academic Press, New York, 1962)
[2] S.W. Hawking, Phys. Rev. D37 (1988) 904
[3] S.B. Giddings and A. Strominger, Nucl. Phys. B306 (1988) 890
[4] S.W. Hawking and R. Laflamme, Phys. Lett. B209 (1988) 39
[5] S. Coleman, Nucl. Phys. B307 (1988) 864
[5a] W. Unruh, Phys. Rev. D40 (1989) 1053
[5b] J. Polchinski, Nucl. Phys. B324 (1989) 123; B325 (1989) 619
[6] A. Lyons, Nucl. Phys. B324 (1989) 253
[7] H.F. Dowker, Nucl. Phys. B331 (1990) 194
[8] J.B. Hartle and S.W. Hawking, Phys. Rev. D28 (1983) 2960
[9] J.S. Dowker and D. Pettengill, University of Manchester report, unpublished (1974)
[10] J.W. York, Phys. Rev. Lett. 28 (1972) 1082
[11] G.W. Gibbons and S.W. Hawking, Phys. Rev. D15 (1977) 2752
[12] J.J. Halliwell and S.W. Hawking, Phys. Rev. D31 (1985) 1777
[13] S. Wada, Nucl. Phys. B276 (1986) 729
[14] E.M. Lifschitz and I.M. Khalatnikov, Adv. Phys. 12 (1963) 185
[15] L.P. Grishchuk, Zh. Eksp. Teor. Fiz. 67 (1974) 825
[16] D.R. Brill and S. Deser, Commun. Math. Phys. 32 (1973) 291
[17] V. Moncrief, Phys. Rev. D18 (1978) 983
[18] D.R. Brill, Nuovo Cim. Suppl., vol. II (1964) 1
[19] J.J. Halliwell and R. Laflamme, Class. Quant. Grav. 6 (1989) 1839
[20] S.W. Hawking and D.N. Page, Phys. Rev. D42 (1990) 2655
[21] N.H. Barth and M. Christensen, Phys. Rev. D28 (1983) 1876
[22] K.S. Stelle, Phys. Rev. D16 (1977) 953
[23] R. Penrose, *in* General relativity; An Einstein centenary survey, ed. S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979)
[24] L.C. Biedenharn, J. Math. Phys. 2 (1961) 433
[25] U.H. Gerlach and U.K. Sengupta, Phys. Rev. D18 (1978) 1773
[26] G.W. Gibbons, Ann. Phys. 125 (1980) 98
[27] B.L. Hu, J. Math. Phys. 15 (1974) 1748
[28] R.T. Jantzen, J. Math. Phys. 19 (1978) 1163